9/23/25

Lecture 9

Partisions

- Disjoint : A and B are disjoint if their intersection is empty
  - ↳ No elements in common

Functions

- $f: A \to B$ is an assignment from A to exactly one element of B.

Injection

- one-to-one function

Surjection (onto)

- all elements in the codomain are mapped

Bijection

- both injection and surjection

$\boxed{\text{Example}}$

$f: \mathbb{R} \to \mathbb{R} \quad f(x) = 2x+1$

Injectivity:

$\forall x \in \mathbb{R}, \quad \forall y \in \mathbb{R} \quad (f(x) = f(y) \to x = y)$

$$2x+1 = 2y+1$$
$$\Rightarrow 2x = 2y$$
$$\Rightarrow x = y$$

Surjectivity

$\forall\ \underset{\substack{\| \\ y}}{f(x)} \in \mathbb{R} \quad \exists x$

$$2x+1 = y$$
$$2x = y-1$$
$$\boxed{x = \frac{y-1}{2}} \qquad y \in \mathbb{R}$$

$f: \mathbb{R} \to \mathbb{R} \quad f(x) = x^2$

$$y = f(x) \in \mathbb{R}$$
$$y = x^2$$
$$\begin{cases} \pm\sqrt{y} = x & \text{when } y \geq 0 \\ \pm\sqrt{y}\, i = x & \text{when } y < 0 \end{cases}$$

Not a Surjection!

Inverse Functions

Def: Let f be a bijection from A to B. Then the inverse of f, denoted f^{-1}, is the function from B to A.

Composition

Def: Let $f: B \to C$, $g: A \to B$. The composition of f with g, denoted $f \circ g$ is the function from A to C.

$\boxed{\text{Example}}$

$$f: \underset{\{1,2,3\}}{\underbrace{[1,3]}} \to \mathbb{Z} \qquad f(x) = x^2 \qquad \text{Range}(f) = \{1, 4, 9\}$$

$$g: \underset{\{1,2,3 \ldots 15,16\}}{\underbrace{[1,16]}} \to \mathbb{Z} \qquad g(x) = x^2$$

So $g \circ f = g(f(x))$ works because

- The RANGE of the INNER FUNCTION has to be a <u>subset</u> of the DOMAIN of the OUTER FUNCTION.

Cardinality of Infinite Sets (Not in Zybooks, WILL BE ON EXAM 2)

- Some infinities are bigger than others:

$$|\mathbb{Z}| < |\mathbb{R}| < |P(\mathbb{R})|$$

- <u>Relative Definitions</u>

1. $|A| = |B|$, i.f.f. there is <u>bijection</u> from A to B
2. $|A| \leq |B|$, if there is an <u>injection</u> from A to B
3. When $|A| \leq |B|$ and $|A| \neq |B|$, we write $|A| < |B|$
4. $|A| < |P(A)|$
5. $A \subseteq B \to |A| \leq |B|$

Example

$D = [0,1]$, $E = [0,2]$

- A bijective function $f(x) = 2x$
- So $|A| = |B|$

- A set is <u>countable</u> if :
  - <u>A is finite</u> OR $|A| \leq |\mathbb{Z}^+|$
  - Basically you want to find a bijection that maps A to $\{1, 2, 3, 4, \ldots\}$, to show A is countable.

- If that is not the case then the set is uncountable.

Countable vs Uncountable

$[-1000, 1000] \cap \mathbb{Z}$ Countable

$\{10, 11, 12 \ldots\}$

Positive Rational Numbers

Definition : A rational number can be expressed as the ratio of two Integers p and q such that $q \neq 0$.

- Positive rational numbers are countable.